# SOME ASYMPTOTIC SOLUTIONS OF THE DIFFUSION EQUATION

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Inzhenerno-Fizicheskii Zhurnal, Vol. 14, No. 1, pp. 108–114, 1968

UDC 536.423.1

The equations describing the variation of the radius of a water droplet with time are investigated for quasi-stationary and nonstationary evaporation processes. Solutions of these equations are found in the form of asymptotic series in powers of a small dimensionless parameter. Some properties of the solutions are determined.

This paper is a continuation of [1], where the nonstationary evaporation of a spherical droplet fixed relative to a homogeneous medium of infinite extent was studied.

By means of an integral transformation of the type

$$\bar{V}=\int\limits_R^\infty V\exp(-\alpha r)\,dr,$$

the problem of the variation of the droplet radius in time (considering the effect of the reduction in droplet size on the rate of the nonstationary process) was reduced to the problem at the moving liquid-gas phase interface. Applying the Laplace transformation a nonlinear integrodifferential equation was obtained for the radius of the evaporating droplet with solution in the form of an asymptotic series in powers of the small dimensionless parameter Ko:

$$\begin{aligned}R&=R_0+\\&+\mathrm{Ko}\left[\frac{D(s-1)}{R_0}\,t+\frac{2\sqrt{D}(s-1)}{\sqrt{\pi}}\sqrt{t}\right]-\\&-\mathrm{Ko}^2\left\{\frac{D^2(s-1)^2}{2R_0^3}t^2+\frac{2D\sqrt{D}(s-1)^2}{3R_0^2\sqrt{\pi}}\,t\sqrt{t}-\right.\\&-\left[\frac{3D(s-1)^2}{2R_0}-\frac{2D(s-1)^2}{R_0\pi}+\frac{2D(s-1)}{R_0}\right]t-\\&\left.-\left[\frac{4\sqrt{D}(s-1)^2}{\pi\sqrt{\pi}}+\frac{4\sqrt{D}(s-1)}{\sqrt{\pi}}\right]\sqrt{t}\right\}+\ldots\end{aligned}\tag{1}$$

If the effect of the decrease in droplet size on the rate of the process is neglected, it is easy to obtain the following relation for the vapor density distribution in the space surrounding the droplet:

$$\rho_c=\rho_c^0+(\rho_s-\rho_c^0)\,\frac{R}{r}\,\mathrm{erfc}\left(\frac{r-R}{2\sqrt{Dt}}\right),\tag{2}$$

where

$$\mathrm{erfc}(z)=1-\mathrm{erf}(z),$$

$$\mathrm{erf}(z)=\frac{2}{\sqrt{\pi}}\int\limits_0^z\exp(-z^2)\,dz.$$

Using (2), we can write the equation of mass transfer across the droplet surface as follows:

$$\frac{dR^2}{dt}=\frac{2D}{\gamma}\,(\rho_c^0-\rho_s)\left(1+\frac{R}{\sqrt{\pi Dt}}\right).\tag{3}$$

We find the solution of nonlinear differential equation (3) in series form:

$$R=r_0+\mathrm{Ko}\,r_1+\mathrm{Ko}^2r_2+\mathrm{Ko}^3r_3+\ldots\tag{4}$$

Substituting expansion (4) into (3) and equating the coefficients of like powers of Ko on the left and right sides, we obtain the following equations for the functions $r_0$, $r_1$, $r_2$, $r_3$, ...:

$$\begin{aligned}&r_0=R_0,\quad \frac{dr_1}{dt}=\frac{D(s-1)}{R_0}+\frac{\sqrt{D}(s-1)}{\sqrt{\pi t}},\\&\frac{dr_2}{dt}=-\frac{D^2(s-1)^2}{R_0^3}\,t-\frac{2D\sqrt{D}(s-1)^2}{R_0^2\sqrt{\pi}}\sqrt{t},\\&\frac{dr_3}{dt}=\frac{3D^3(s-1)^3}{2R_0^5}t^2+\frac{16D^2\sqrt{D}(s-1)^3}{3R_0^4\sqrt{\pi}}\,t\sqrt{t}+\\&\qquad+\frac{4D^2(s-1)^3}{R_0^3\pi}\,t,\ \ldots\end{aligned}\tag{5}$$

Using the solutions of Eqs. (5), we write expansion (4) as

$$\begin{aligned}R&=R_0+\\&+\mathrm{Ko}\left[\frac{D(s-1)}{R_0}\,t+\frac{2\sqrt{D}(s-1)}{\sqrt{\pi}}\sqrt{t}\right]-\\&-\mathrm{Ko}^2\left[\frac{D^2(s-1)^2}{2R_0^3}\,t^2+\frac{4D\sqrt{D}(s-1)^2}{3R_0^2\sqrt{\pi}}\,t\sqrt{t}\right]+\\&+\mathrm{Ko}^3\left[\frac{D^3(s-1)^3}{2R_0^5}\,t^3+\frac{32D^2\sqrt{D}(s-1)^3}{15R_0^4\sqrt{\pi}}\,t^2\sqrt{t}+\right.\\&\qquad\left.+\frac{2D^2(s-1)^3}{R_0^3\pi}\,t^2\right]-\ldots\end{aligned}\tag{6}$$

The differential equation (3) can also be solved by successive approximations using the recurrence relation

$$\frac{dR_n}{dt}=D\,\mathrm{Ko}\,(s-1)\left(\frac{1}{R_{n-1}}+\frac{1}{\sqrt{\pi Dt}}\right)$$

$$(n=1,\ 2,\ 3,\ \ldots).$$

It is easy to show that by selecting the initial radius of the evaporating droplet as the zero-order approximation, we obtain a first approximation of the solution that coincides with the first two terms of series (6), etc.

In the case of steady-state evaporation the mass flux through any spherical surface concentric with the droplet is a constant quantity given by

$$I=-4\pi r^2D\,\frac{\partial\rho_c}{\partial r}.\tag{7}$$

In this case

$$\rho_c=\rho_c^0+\frac{R}{r}\,(\rho_s-\rho_c^0).\tag{8}$$

Since the droplet radius and hence the rate of evaporation continuously decrease, the evaporation process cannot be stationary. However, if the saturated vapor density at the droplet temperature is much less than the droplet density, evaporation may be assumed quasi-stationary [2].

Since $I = -dm/dt$, where $m = 4\pi R^3\gamma/3$ is the mass of the droplet, we can write Eq. (7) with distribution (8) in the form

$$R\frac{dR}{dt} = \mathrm{Ko}\,D(s-1). \tag{9}$$

We find the solution in the form of series (4), which, in this case, can be written as follows:

$$R = R_0 + \mathrm{Ko}\,\frac{D(s-1)}{R_0}t - \mathrm{Ko}^2\frac{D^2(s-1)^2}{2R_0^3}t^2 + \mathrm{Ko}^3\frac{D^3(s-1)^3}{2R_0^5}t^3 - \ldots. \tag{10}$$

After multiplying the asymptotic series by itself, we have

$$R^2 = R_0^2 + 2\mathrm{Ko}\,D(s-1)t,$$

which is also easily obtained by direct integration of Eq. (9).

To simplify, we transform in (1), (6), and (10) to the dimensionless variables

$$\nu = R/R_0, \quad \tau = t/t_0.$$

Selecting the characteristic time $t_0 = R_0^2/D$, we have

$$\nu = 1 + \mathrm{Ko}(s-1)\tau - \mathrm{Ko}^2\frac{(s-1)^2}{2}\tau^2 + \mathrm{Ko}^3\frac{(s-1)^3}{2}\tau^3 - \ldots,$$

$$\nu^* = 1 + \mathrm{Ko}\left[(s-1)\tau + \frac{2(s-1)}{\sqrt{\pi}}\sqrt{\tau}\right] - \mathrm{Ko}^2\left[\frac{(s-1)^2}{2}\tau^2 + \frac{4(s-1)^2}{3\sqrt{\pi}}\tau\sqrt{\tau}\right] + \mathrm{Ko}^3\left[\frac{(s-1)^3}{2}\tau^3 + \frac{32(s-1)^3}{15\sqrt{\pi}}\tau^2\sqrt{\tau} + \frac{2(s-1)^3}{\pi}\tau^2\right] - \ldots,$$

$$\nu^{**} = 1 + \mathrm{Ko}\left[(s-1)\tau + \frac{2(s-1)}{\sqrt{\pi}}\sqrt{\tau}\right] - \mathrm{Ko}^2\left[\frac{(s-1)^2}{2}\tau^2 + \frac{2(s-1)^2}{3\sqrt{\pi}}\tau\sqrt{\tau} - \frac{3(s-1)^2}{2}\tau + \frac{2(s-1)^2}{\pi}\tau - 2(s-1)\tau - \frac{4(s-1)^2}{\pi\sqrt{\pi}}\sqrt{\tau} - \frac{4(s-1)}{\sqrt{\pi}}\sqrt{\tau}\right] + \ldots. \tag{11}$$

From the first two relations in (11) it is easy to obtain

$$\nu - \nu^* = \mathrm{Ko}\frac{2(1-s)}{\sqrt{\pi}}\sqrt{\tau}\left\{1 + \frac{2}{3}\mathrm{Ko}(1-s)\tau + \mathrm{Ko}^2\left[\frac{16(1-s)^2}{15}\tau^2 + \frac{(1-s)^2}{\sqrt{\pi}}\tau\sqrt{\tau}\right] + \ldots\right\}, \tag{12}$$

$$\frac{d\nu}{d\tau} - \frac{d\nu^*}{d\tau} = \mathrm{Ko}\frac{1-s}{\sqrt{\pi\tau}}\left\{1 + 2\mathrm{Ko}(1-s)\tau + \mathrm{Ko}^2\left[\frac{4(1-s)^2}{\sqrt{\pi}}\tau\sqrt{\tau} + \frac{16(1-s)^2}{3}\tau^2\right] + \ldots\right\}. \tag{13}$$

Since when $s < 1$ (the droplet evaporates) the right sides of (12) and (13) are positive, $\nu > \nu^*$, $d\nu/dt > d\nu^*/d\tau$ or in dimensional variables

$$R > R^*, \tag{14}$$

$$\frac{dR}{dt} > \frac{dR^*}{dt}. \tag{15}$$

Inequality (14) shows that at all values of $\tau$ on the interval $0 < \tau < \tau_{te}$ ($\tau_{te}$ is the dimensionless total evaporation time in the quasi-stationary process) the $R = R(t)$ curve lies above the $R^* = R^*(t)$ curve.

Since when $s < 1$ $dR/dt$ and $dR^*/dt$ are negative, we can write (15) as

$$\left|\frac{dR^*}{dt}\right| > \left|\frac{dR}{dt}\right|,$$

it follows that the absolute value of the rate of decrease of the droplet radius in the nonstationary evaporation process is greater than the absolute value of the corresponding rate in the quasi-stationary process.

From the first and third of Eqs. (11) we derive

$$\nu - \nu^{**} = \mathrm{Ko}\frac{2(1-s)}{\sqrt{\pi}}\sqrt{\tau}\left\{1 + \mathrm{Ko}\frac{1-s}{3}\tau + \frac{1}{2}\mathrm{Ko}\left[\frac{2(1-s)}{\pi} + 2 + \frac{3(s-1)}{2}\right]\sqrt{\pi\tau} + 2\mathrm{Ko}\left[1 + \frac{s-1}{\pi}\right] + \ldots\right\}, \tag{16}$$

$$\frac{d\nu}{d\tau} - \frac{d\nu^{**}}{d\tau} = \mathrm{Ko}\frac{1-s}{\sqrt{\pi\tau}}\left\{1 + \mathrm{Ko}(1-s)\tau + \mathrm{Ko}\left[\frac{2(1-s)}{\pi} + 2 + \frac{3(s-1)}{2}\right]\sqrt{\pi\tau} + 2\mathrm{Ko}\left[1 + \frac{s-1}{\pi}\right] + \ldots\right\}. \tag{17}$$

Since when $s < 1$ the right sides of (16) and (17) are positive,

$$\nu > \nu^{**}, \quad \frac{d\nu}{d\tau} > \frac{d\nu^{**}}{d\tau}$$

or in dimensional variables

$$R > R^{**}, \tag{18}$$

$$\frac{dR}{dt} > \frac{dR^{**}}{dt}. \tag{19}$$

From (18) and (19) there follow the same physical conclusions as from inequalities (14) and (15).

An analysis of (12), (13), (16), and (17) yields as $t \to 0$ the asymptotic formulas

$$\frac{dR^*}{dt} \approx \frac{dR}{dt} + \mathrm{Ko}\,\frac{\sqrt{D}(s-1)}{\sqrt{\pi t}},$$

$$R^* \approx R + \mathrm{Ko}\,\frac{2\sqrt{D}(s-1)}{\sqrt{\pi}}\sqrt{t},$$

$$\frac{dR^{**}}{dt} = \frac{dR}{dt} + \mathrm{Ko}\,\frac{\sqrt{D}(s-1)}{\sqrt{\pi t}} \times$$

$$\times \left\{1 + 2\mathrm{Ko}\left[1 + \frac{s-1}{\pi}\right]\right\} + O(B),$$

$$R^{**} = R^* + O(F\sqrt{t}).$$

It should be kept in mind that B and F are of the order of $\mathrm{Ko}^2$, while when $s < 1$, $B > 0$, $F < 0$.

With (6) and (10), it can be shown that for the difference $R^2 - (R^*)^2$ there is an expansion of the form

$$R^2 - (R^*)^2 = \mathrm{Ko}\,\frac{4\sqrt{D}(1-s)R_0}{\sqrt{\pi}}\sqrt{t} -$$

$$-\mathrm{Ko}^2\left[\frac{4D\sqrt{D}(1-s)^2}{3R_0\sqrt{\pi}}\,t\sqrt{t} + \frac{4D(1-s)^2}{\pi}\,t\right] -$$

$$-\mathrm{Ko}^3\left[\frac{2D^2(1-s)^3}{5R_0^3\sqrt{\pi}}\,t^2\sqrt{t} +\right.$$

$$\left. + \frac{4D^2(1-s)^3}{3R_0^2\pi}\,t^2\right] - \ldots.$$

After multiplying the left and right sides of this equation by $4\pi$, we obtain

$$\sigma - \sigma^* = 16\mathrm{Ko}(1-s)R_0\sqrt{\pi D t} + O(t) \quad (t \to 0),$$

$$\frac{d\sigma}{dt} - \frac{d\sigma^*}{dt} =$$

$$= \frac{8\sqrt{\pi D}\,\mathrm{Ko}(1-s)}{\sqrt{t}}\,R_0 + O(A) \quad (t \to 0).$$

From these relations it follows that as $t \to 0$ and $s < 1$ (the droplet evaporates) $\sigma > \sigma^*$; the difference between $\sigma$ and $\sigma^*$ increases with increase in the initial droplet radius; when $s > 1$ (condensation takes place at the droplet surface) $\sigma^* > \sigma$; consequently, at the beginning of the process the $\sigma^* = \sigma^*(t)$ curve lies above the $\sigma = \sigma(t)$ curve; the absolute rate of variation of the surface for the evaporating droplet is less for the quasi-stationary than for the nonstationary process.

The properties of the curve representing the function $R = R(t)$ in both the quasi-stationary and nonstationary cases are of considerable interest, particularly from the theoretical standpoint. We will consider some of these properties.

Since in the quasi-stationary process for any t

$$\frac{dR}{dt} = -\frac{\mathrm{Ko}\,D(1-s)}{R} < 0,$$

$$\frac{d^2R}{dt^2} = -\frac{\mathrm{Ko}^2D^2(1-s)^2}{R^3} < 0,$$

the graph of the function $R = R(t)$ is a convex curve with a gradient decreasing from $-\mathrm{Ko}D(1-s)/R_0$ at $t = 0$ to $-\infty$ at $t = t_{te}$.

When $s > 1$ (condensation takes place at the droplet surface) $R = R(t)$ is an increasing function. Its graph is a convex curve with gradient equal to $D\mathrm{Ko}(s-1)/R_0$ at $t = 0$ and tending to zero as $t \to +\infty$.

After differentiation with respect to t we write relation (3) as

$$\frac{d^2R^*}{dt^2} = \mathrm{Ko}\,D(s-1) \times$$

$$\times\left[-\frac{1}{(R^*)^2}\,\frac{dR^*}{dt} - \frac{1}{2t\sqrt{\pi D t}}\right]. \tag{20}$$

Since when $s < 1$, $dR^*/dt < 0$, the second derivative of the function $R^* = R^*(t)$ must change sign at some value of $t_1$. Consequently, in the case of nonstationary evaporation the function $R^* = R^*(t)$ has a point of inflection, for determining which we obtain the following approximate algebraic equation after a number of transformations.

$$2D\,\mathrm{Ko}(1-s)t_1(\sqrt{\pi D t_1} + R_0) = R_0^3. \tag{21}$$

From the physical standpoint only pairs of positive numbers $t_1$ and $R_0$ are of interest (meaningful).

An analysis of Eq. (21) shows that as $R_0$ decreases the time corresponding to the point of inflection on $R^* = R^*(t)$ also decreases, i.e., $t_1 = \varphi(R_0)$ obtained from (21) is an increasing function.

Since $\lim\limits_{t \to +0} \frac{dR^*}{dt} = -\infty$, $R^* = R^*(t)$—which describes the variation of the droplet radius in time in the nonstationary evaporation process without allowance for the effect of the reduction of droplet size on the rate of the process—is concave at $0 < t < t_1$ and convex at $t > t_1$.

Equation (20) shows that in the case of a nonstationary condensation process $d^2R^*/dt^2 < 0$ at any t. Consequently, $R^* = R^*(t)$ is an increasing function, whose graph is a curve convex on the entire time axis with a gradient equal to $D\mathrm{Ko}(s-1)/R_0$ at $t = 0$ and tending to zero as $t \to +\infty$.

For a nonstationary evaporation process with allowance for the effect of the reduction in droplet size on the rate of the process the $R^{**} = R^{**}(t)$ curve has the same properties as the $R^* = R^*(t)$ curve.

## NOTATION

R is the droplet radius; t is the time; $\alpha$ is a constant satisfying the condition $\mathrm{Re}\,\alpha > 0$; Ko is a dimensionless parameter equal to the ratio of the saturated vapor density at the droplet temperature to the droplet density; D is the diffusion coefficient for vapor in air; s is the supersaturation; $R_0$ is the initial droplet radius; $\rho_c$ is the vapor density in air; $\rho_c^0$ is its initial value; $\rho_s$ is the saturated vapor density at droplet temperature; r is a space coordinate; $\gamma$ is the density of water; I is the mass flux across a spherical surface of radius r; $\nu^*$ and $R^*$ are the dimensionless and dimensional droplet radius in the nonstationary process of evaporation without allowance for the effect of reduction in droplet size on the rate of the process, respectively; $\nu^{**}$ and $R^{**}$ are the same with allowance for the effect of droplet size on the rate of the process; $\sigma$ and $\sigma^*$ are the surfaces of droplets with radii R and $R^*$, respectively; A is a constant equal to $16DKo^2(1 - s)^2$; $t_{te}$ is the total droplet evaporation time; $t_1$ is the time corresponding to the point of inflection on the $R^* = R^*(t)$ curve.

## REFERENCES

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